

MINKOWSKI SUM OF A VORONOI PARALLELOTOPE AND A SEGMENT

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ABSTRACT. By a *Voronoi parallelootope* $P(a)$ we mean a parallelootope determined by a non-negative quadratic form a . It was studied by Voronoi in his famous memoir. For a set of vectors \mathcal{P} , we call its *dual* a set of vectors \mathcal{P}^* such that $\langle p, q \rangle \in \{0, \pm 1\}$ for all $p \in \mathcal{P}$ and $q \in \mathcal{P}^*$. We prove that Minkowski sum of a Voronoi parallelootope $P(a)$ and a segment is a Voronoi parallelootope $P(a + a_e)$ if and only if this segment is parallel to a vector e of the dual of the set of normal vectors of all facets of $P(a)$, where $a_e(p) = b\langle e, p \rangle^2$ is a quadratic form of rank 1 related to the segment.

1. INTRODUCTION

Consider a centrally symmetric d -polytope $P(a)$ given by the following system of inequalities

$$(1) \quad P(a) = \{x \in \mathbb{R}^d : \langle p, x \rangle \leq a(p) \text{ for all } p \in \mathcal{P}\},$$

where $\langle p, x \rangle$ is a scalar product of vectors $p, x \in \mathbb{R}^d$. Here $\mathcal{P} \subset \mathbb{R}^d$ is a symmetric set of vectors containing *normal* vectors of all facets of $P(a)$, where *symmetric* means that if $p \in \mathcal{P}$ then $-p \in \mathcal{P}$, too. The function $a : \mathcal{P} \rightarrow \mathbb{R}$ is an arbitrary function.

The above d -dimensional polytope $P(a)$ is called a *Voronoi parallelootope* if the following conditions hold:

- (i) the function $a(p) = \langle p, Ap \rangle$ is a non-negative quadratic form;
- (ii) the set \mathcal{P} contains the set $\mathcal{P}_s(a) \subset L$ of normal vectors of all facets of the polytope $P(a)$;
- (iii) the set $\mathcal{P}_s(a)$ generates integrally a d -dimensional lattice L .

Recall that a *parallelootope* is a polytope whose parallel translations fill its space without interstices (gaps) and intersections by inner points. Voronoi proved in [Vo1908] that if the above conditions (i), (ii) and (iii) hold, then $P(a)$ is a parallelootope. Besides, the parallelootope $P(a)$ is a Dirichlet-Voronoi cell of the lattice $2AL$ with respect to the metric form a .

One can prove that the set \mathcal{P} can be enlarged up to a set $\mathcal{P}(a) \subset L$ of minimal (with respect to the form a) vectors of each parity class of L . Moreover, the set \mathcal{P} may be the whole lattice L .

Let $q \in \mathbb{R}^d$ be a vector and $\alpha \in \mathbb{R}$ be a number. Define the following affine hyperplanes

$$(2) \quad H(q, \alpha) = \{x \in \mathbb{R}^d : \langle q, x \rangle = \alpha\} \text{ and } H_p(a) = H(p, a(p)).$$

Note that only for $p \in \mathcal{P}(a)$ the hyperplane $H_p(a)$ supports the Voronoi parallelootope $P(a)$ at a face $F(p)$. This face $F(p)$ is called *contact face* of $P(a)$. Hence, we call vectors $p \in \mathcal{P}(a)$ by *contact vectors*. Dolbilin call in [Do09] contact faces by *standard faces*.

For each $p \in \mathcal{P}(a)$, the vector $2Ap$ is called *commensurate* (with the parallelootope $P(a)$). The commensurate vector $2Ap$ connects the center of the parallelootope $P(a)$ with the center of a parallelootope that is adjacent to $P(a)$ by the contact face $F(p)$. Commensurate vectors generate the lattice $2AL$.

Recall that the dual of a lattice L is

$$L^* = \{q \in \mathbb{R}^d : \langle q, p \rangle \in \mathbb{Z} \text{ for all } p \in L\}.$$

Since the set $\mathcal{P}_s(a)$ generate the above lattice L , we can change L by $\mathcal{P}_s(a)$ in the above definition of L^* . Define the following important subset $\mathcal{P}_s^*(a) \subset L^*$ as follows

$$(3) \quad \mathcal{P}_s^*(a) = \{e \in \mathbb{R}^d : \langle e, p \rangle \in \{0, \pm 1\} \text{ for all } p \in \mathcal{P}_s(a)\}.$$

We call this set *dual* of $\mathcal{P}_s(a)$. Each vector $e \in \mathcal{P}_s^*(a)$ determines a partition of the lattice L onto $(d-1)$ -dimensional layers.

Lemma 1. *Let $e \in \mathcal{P}_s^*(a)$. Then*

$$L = \cup_{z \in \mathbb{Z}} L_e(z),$$

where

$$(4) \quad L_e(z) = L \cap H(e, z)$$

is a $(d-1)$ -dimensional layer of L , and the hyperplane $H(e, z)$ is defined in (2).

Proof. Since the set of normal vectors $\mathcal{P}_s(a)$ generates the lattice L , for any $v \in L$, we have $v = \sum_{p \in \mathcal{P}_s(a)} z_p p$, where $z_p \in \mathbb{Z}$. Hence

$$\langle e, v \rangle = \sum_{p \in \mathcal{P}_s(a)} z_p \langle e, p \rangle = z \in \mathbb{Z},$$

i.e. $v \in L_e(z)$. Since $0 \in L_e(0) \subset L$, $L_e(z) = v + L_e(0)$. Since L is a d -dimensional lattice, each layer has dimension $d-1$. \square

Let $e \in \mathbb{R}^d$ be a vector and $l(e)$ be a line spanned by e . Let

$$(5) \quad z(e) = \{x \in \mathbb{R}^d : x = \lambda e, \quad -1 \leq \lambda \leq 1\}.$$

be a segment of the line $l(e)$ symmetric with respect origin 0. For the vector e and a number $b > 0$, define the following quadratic form of rank 1

$$(6) \quad a_e(p) = b \langle p, e \rangle^2,$$

We prove below, that $bz(e) = P(a_e)$ if $\langle p, e \rangle \in \{0, \pm 1\}$ for all $p \in \mathcal{P}$.

A parallelotope P is called *reducible* if $P = P_1 \oplus P_2$, where \oplus denotes direct sum. Otherwise, P is called *irreducible*.

In this paper we prove the following

Theorem 1. *Let $P(a)$ be an irreducible Voronoi parallelotope, defined in (1), where $\mathcal{P} \supseteq \mathcal{P}(a) \supseteq \mathcal{P}_s(a)$. Let $e \in \mathbb{R}^d$ be a vector. Then one can choose a length of the vector e such that the following assertions are equivalent:*

(i) *Minkowski sum $P(a) + bz(e)$ is a Voronoi parallelotope for any $b \geq 0$, and*

$$P(a) + bz(e) = P(a) + P(a_e) = P(a + a_e);$$

(ii) *$e \in \mathcal{P}_s^*(a)$;*

The implication (ii) \Rightarrow (i) was proved in [Gr06].

Magazinov proved in [Ma13] that (in our terms) $P(a) + bz(e)$ is a Voronoi parallelotope if this sum is a parallelotope. It seems to us that our proof is simpler.

2. MINKOWSKI SUM OF POLYTOPES

For a fixed set \mathcal{P} of normal vectors, the polytopes $P(a)$ defined in (1) have the following simple property

Lemma 2. *For any functions $a_1(p)$ and $a_2(p)$, the following inclusion holds.*

$$P(a_1) + P(a_2) \subseteq P(a_1 + a_2).$$

Proof. For $k \in \{1, 2\}$, let $x_k \in P(a_k)$. Then $\langle p, x_k \rangle \leq a_k(p)$ for all $p \in \mathcal{P}$. This implies that $\langle p, (x_1 + x_2) \rangle \leq a_1(p) + a_2(p)$ for all $p \in \mathcal{P}$, i.e., $x_1 + x_2 \in P(a_1 + a_2)$. Hence $P(a_1) + P(a_2) \subseteq P(a_1 + a_2)$. \square

Let a set \mathcal{P} of normal vectors be fixed. It is a problem to find conditions when the equality $P(a_1) + P(a_2) = P(a_1 + a_2)$ holds. We show below that this equality holds when $P(a_2)$ is a segment $bz(e)$ and $a_2 = f_e$, where the function $f_e(p)$ is defined below in (7).

For $i = 1, 2$, let a_i be a non-negative quadratic form and $P(a_i)$ be the corresponding Voronoi parallelootope described in (1). It is also a problem to find conditions when the sum $P(a_1) + P(a_2)$ is a parallelootope, and, in particular, it is a Voronoi parallelootope.

It is shown in [RBo05] that the equality $P(a_1) + P(a_2) = P(a_1 + a_2)$ holds if a_1 and a_2 belong to closure of an L-type domain. We show below that this equality holds if $a_2(p) = a_e(p)$, where the quadratic form $a_e = b\langle p, e \rangle^2$ of rank 1 relates to a segment $bz(e)$, and then the sum $P(a_1) + P(a_e)$ is a parallelootope.

3. SEGMENTS

Let $e, p \in \mathbb{R}^d$ be some vectors. Consider the affine hyperplane $H_p(f_e)$ defined in (2), where

$$(7) \quad f_e(p) = b \frac{\langle p, e \rangle^2}{|\langle p, e \rangle|}.$$

Here b is a non-negative weight of the segment $z(e)$ defined in (5). It is natural to suppose that $f_e(p) = 0$ if $\langle p, e \rangle = 0$.

Lemma 3. *For any vector $p \in \mathbb{R}^d$, the hyperplane $H_p(f_e)$ supports the segment $bz(e)$.*

Proof. Note that end-vertices of the segment $bz(e)$ are points $\pm be$. If $\langle p, e \rangle > 0$, then the end-vertex be lies on $H_p(f_e)$. If $\langle p, e \rangle < 0$, then the end-vertex $-be$ lies on $H_p(f_e)$. If $\langle p, e \rangle = 0$, then the whole segment $bz(e)$ lies on $H_p(f_e)$. \square

Lemma 3 implies the following fact.

Lemma 4. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a set of vectors such that scalar products $\langle p, e \rangle$ have all the three signs $+$, $-$ and 0 . Let $P(a)$ be given by (1). Then*

$$bz(e) = P(f_e),$$

where the function $f_e(p)$ is defined in (7). \square

4. MINKOWSKI SUM OF A POLYTOPE WITH A SEGMENT

At first, we consider the Minkowski sum $P(a) + z(e)$ of an *arbitrary polytope* $P(a)$ defined in (1) and the segment $z(e)$ defined in (5). Á.Horváth call in [Ho07] the sum $P + z(e)$ by an *extension* P^e of P . So, we consider a polytope $P = P(a)$ described by the inequalities in (1), where $a = a(p)$ is an arbitrary function defined on a symmetric set \mathcal{P} . Recall that we call a face F contact and denote it by $F(p)$ if $F = P(a) \cap H_p(a)$. The vector p is called contact vector of the face $F = F(p)$.

For a face F of a polytope $P = P(a)$, let $l_F(e)$ be a parallel shift of the line $l(e)$ such that $l_F(e) \cap F \neq \emptyset$. Call the face F *transversal* to e if $l_F(e) \cap F$ is a point. Otherwise, call the face F *parallel* to e and denote this fact as $F \parallel e$.

We say that a face F belongs to a *shadow boundary* of P in direction e if $l_F(e) \cap F = l_F(e) \cap P$. Denote by $\mathcal{F}_e(P)$ a set of all faces of P that belong to the shadow boundary of P in direction of e .

Note that the face F is transformed into a face $F + z(e)$ in the extension $P^e = P + z(e)$. Denote dimension of F by $\dim F$. Lemma 5 below helps to understand how change faces of P^e with respect to faces of P . Assertions of Lemma 5 are obvious.

Lemma 5. *Let F be a face of a polytope P . Consider the sum $P^e = P + z(e)$. There are the following three possibilities for the sum $F + z(e)$:*

- (i) *if F is parallel to e , then $F + z(e) = F^e$ is an extension of F , and $\dim(F + z(e)) = \dim F$;*
- (ii) *if F is transversal to e and $F \notin \mathcal{F}_e(P)$, then $F + z(e)$ is a parallel shift of F ;*
- (iii) *if F is transversal to e and $F \in \mathcal{F}_e(P)$, then $F + z(e)$ is direct sum of F and $z(e)$, and $\dim(F + z(e)) = \dim F + 1$. \square*

According to Lemma 5, each *facet* F of the sum $P + bz(e)$ has one of the following three types

- (i) extension $F = F_1^e$ of a facet F_1 of P ;
- (ii) a parallel shift $F = F_1 + bz(e)$ of a facet F_1 of P ;
- (iii) direct sum $F = G \oplus bz(e)$ of a $(d - 2)$ -face G of P and the segment $bz(e)$.

Now consider Minkowski sum of a polytope $P(a)$ given in (1) and a segment $bz(e)$. Suppose that signs of $\langle p, e \rangle$ take all three values of the set $\{0, \pm 1\}$ for all $p \in \mathcal{P}$. Then, by Lemma 4, $bz(e) = P(f_e)$, where the function $f_e(p)$ is defined in (7). Obviously, the polytope $P(a)$ is supported by the hyperplane $H_p(a)$ defined in (2) at every *facet* of P .

Proposition 1. *Let $P = P(a)$ be a polytope described in (1). Suppose that each $(d - 2)$ -face $F \in \mathcal{F}_e(P)$, which is transversal to e , is a contact face $F = F(p)$ such that $\langle e, p \rangle = 0$. Then the following equalities hold*

$$P(a) + bz(e) = P(a) + P(f_e) = P(a + f_e).$$

Proof. We show that each facet $F_e(p)$ of the polytope $P^e = P(a) + bz(e)$ is supported by the hyperplane $H_p(a + f_e)$.

By Lemma 3, the hyperplane $H_p(f_e)$ supports the segment $bz(e)$ for any $p \in \mathcal{P}$. Similarly, $H_p(a)$ supports $P(a)$ if p is a normal vector of a facet of $P(a)$. Let $F(p)$ be a contact face of $P(a)$. Consider the above three cases (i), (ii) and (iii) of facets of the sum $F(p) + bz(e)$.

Case (i). Let $F(p) \in \mathcal{F}_e(P)$ be a facet. Then $e \parallel F(p)$ and therefore $\langle p, e \rangle = 0$. Hence $f_e(p) = 0$, and $H_p(a + f_e) = H_p(a)$ supports the facet $F_e(p) = F(p) + bz(e) = F^e(p)$ of the sum $P + bz(e)$.

Case (ii). Let $F(p)$ be a facet and $F(p) \notin \mathcal{F}_e(P)$. Then $\langle p, e \rangle \neq 0$. Hence the sum $F(p) + bz(e)$ is a shift of $F(p)$ obtained as follows. Let $x \in F(p)$. Then the point

$$x + be \frac{\langle p, e \rangle}{|\langle p, e \rangle|}$$

belongs to $F(p) + bz(e)$. Here the multiple $\frac{\langle p, e \rangle}{|\langle p, e \rangle|}$ describes direction of the shift. Since $F(p)$ is a facet of $P(a)$, we have $\langle p, x \rangle = a(p)$ and therefore

$$\langle p, x + be \frac{\langle p, e \rangle}{|\langle p, e \rangle|} \rangle = a(p) + f_e(p) = (a + f_e)(p).$$

Since x is an arbitrary point of $F(p)$, this implies that the facet $F_e(p) = F(p) + bz(e)$ of $P + bz(e)$ is supported by $H_p(a + f_e)$.

Case (iii). Now, let $F(p)$ be a $(d-2)$ -face of $P = P(a)$, $F(p) \in \mathcal{F}_e(P)$ and $F(p)$ is transversal to e . Then we have the case (iii) of Lemma 5. The face $F(p)$ is transformed into the facet $F_e(p) = F(p) \oplus bz(e)$ of $P + bz(e)$. Since $F(p) \in \mathcal{F}_e(P)$, we have $\langle p, e \rangle = 0$. Hence $f_e(p) = 0$ and the hyperplane $H_p(a + f_e) = H_p(a)$ supports the facet $F_e(p) = F(p) \oplus bz(e)$.

So, each facet of the sum $P + bz(e)$ is supported by a hyperplane $H_p(a + f_e)$. Hence, $P(a) + bz(e) \supseteq P(a + f_e)$. Since, by Lemma 4, $bz(e) = P(f_e)$, according to Lemma 2, we obtain assertion of this Proposition. \square

5. MINKOWSKI SUM OF A VORONOI PARALLELOTOPE WITH A SEGMENT

Now consider Minkowski sum of a *Voronoi parallelootope* $P(a)$ and the segment $bz(e)$. Without loss of generality, we can suppose that \mathcal{P} contains the set of contact vectors of all contact faces of $P(a)$.

Obviously, each segment is a parallelootope, and moreover a Voronoi parallelootope. In fact, we can choose lengths of vectors $p \in \mathcal{P}$ such that $\langle p, e \rangle \in \{0, \pm 1\}$. In this case the function $f_e(p)$ transforms in the quadratic form $a_e(p)$ defined in (6), and $P(a_e)$ is a Voronoi parallelootope.

Let $\mathcal{P}_s(a) \subseteq \mathcal{P}(a)$ be a set of contact vectors of facets of $P(a)$. They are normal vectors of facets. If, for a vector $e \in \mathbb{R}^d$, the inclusions $\langle p, e \rangle \in \{0, \pm 1\}$ hold for all $p \in \mathcal{P}_s(a)$, then one can change a value of b and the length of e such that the length of the segment $bz(e)$ does not change and $\langle p, e \rangle \in \{0, \pm 1\}$ for all $p \in \mathcal{P}_s(a)$. Hence we will consider vectors $e \in \mathcal{P}_s^*(a)$, where the dual $\mathcal{P}_s^*(a)$ is defined in (3). Of course, there may be another vectors $p \in \mathcal{P}$ with $\langle p, e \rangle \in \{0, \pm 1\}$. Hence we introduce the following set

$$(8) \quad \mathcal{P}_e = \{p \in \mathcal{P} : \langle p, e \rangle \in \{0, \pm 1\}\}.$$

Lemma 6. *Let $e \in \mathcal{P}_s^*(a)$, and scalar products $\langle p, e \rangle$ take all three values $0, +1, -1$ for $p \in \mathcal{P}_e$. Then*

$$bz(e) = P(a_e),$$

where the quadratic form a_e is defined in (6), and $P(a)$ is defined in (1).

Proof. It is easy to see that $f_e(p) = a_e(p)$ for all $p \in \mathcal{P}_e$. By Lemma 3, the hyperplane $H_p(a_e)$ supports the segment $bz(e)$ for all $p \in \mathcal{P}_e$. By Lemma 4, $bz(e) = P(a_e)$. \square

Lemma 7. *Let $\mathcal{P}_s(a)$ be a set of normal vectors of a Voronoi parallelootope $P = P(a)$. Let $e \in \mathcal{P}_s^*(a)$, and let $F \in \mathcal{F}_e(P)$ be a $(d-2)$ -face of P that is transversal to e . Then $F = F(p)$ is a contact face for a contact vector p such that $\langle p, e \rangle = 0$.*

Proof. Suppose to the contrary that F generates a 6-belt B . Let $\pm p_1, \pm p_2, \pm p_3 \in \mathcal{P}_s(a)$ be normal vectors of the 6-belt B . Let $F = F(p_1) \cap F(p_2)$. Note that $F(p_1), F(p_2) \notin \mathcal{F}_e(P)$, since F is transversal to e and $F \in \mathcal{F}_e(P)$. Hence, for $i = 1, 2$, $\langle p_i, e \rangle \neq 0$, and therefore $\langle p_i, e \rangle \in \{\pm 1\}$. Since $e \in \mathcal{P}_s^*(a)$, without loss of generality, we can suppose that $\langle p_1, e \rangle = 1$ and $\langle p_2, e \rangle = -1$. Let $F(p_3) \neq F(p_2)$ be the second facet of the 6-belt B that is adjacent to $F(p_1)$. Since $P = P(a)$ is a Voronoi parallelootope, the equality $p_3 = p_1 - p_2$ holds. This equality implies the equality $\langle p_3, e \rangle = \langle p_1, e \rangle - \langle p_2, e \rangle = 2$ that contradicts to $\langle p_3, e \rangle \in \{0, \pm 1\}$. Hence F cannot generate a 6-belt. Therefore $F = F(p)$ is a contact face.

Obviously, $F(p) = F(p_1) \cap F(p_2)$, where $F(p_1), F(p_2)$ are facets of $P(a)$. Then $p = p_1 + p_2$. Since $e \in \mathcal{P}_s^*(a)$, $\langle p_1, e \rangle, \langle p_2, e \rangle \in \{0, \pm 1\}$. If $\langle p_1, e \rangle = \langle p_2, e \rangle = 0$, then $\langle p, e \rangle = 0$. Otherwise, without loss of generality, we can suppose that $\langle p_1, e \rangle = 1, \langle p_2, e \rangle = -1$. this implies $\langle p, e \rangle = 0$. \square

Proposition 2. *Let $P = P(a)$ be a Voronoi parallelootope defined in (1), where $\mathcal{P} \supseteq \mathcal{P}_e \supseteq \mathcal{P}_s(a)$. Then*

$$P(a) + bz(e) = P(a) + P(a_e) = P(a + a_e).$$

Proof. Let $F(p) \in \mathcal{F}_e(P)$ be a $(d-2)$ -face that is transversal to e . Then, by Lemma 7, $F = F(p)$ is a contact face of $P = P(a)$ and $\langle p, e \rangle = 0$. Since $a_e(p) = f_e(p)$ for all $p \in \mathcal{P}_e$, we can apply Proposition 1. Hence the assertion of this Proposition holds. \square

So, we have proved that $P(a) + P(a_e) = P(a + a_e)$ if $e \in \mathcal{P}_s^*(a)$. Recall that $a_e(p) = b\langle p, e \rangle^2$. Now we will prove that if $P(a)$ is irreducible and the sum $P(a) + P(a_e)$ is a parallelotope, then one can choose the length of e such that $e \in \mathcal{P}_s^*(a)$.

Let $G(P)$ be a graph whose vertices correspond to facets of P . Let $v(F)$ be a vertex of $G(P)$ related to a facet F . Two vertices $v(F_1)$ and $v(F_2)$ are adjacent in $G(P)$ if and only if $F_1 \cap F_2$ is a $(d-2)$ -face that generates a 6-belt of P . It is proved in [Or05] that the graph $G(P)$ is connected if and only if the parallelotope P is irreducible. In particular, this means that, for each $p \in \mathcal{P}_s(a)$, the facet $F(p)$ belongs to a 6-belt.

Lemma 8. *Let $P = P(a)$ be an irreducible parallelotope. Let $e \in \mathbb{R}^d$ be a vector such the sum $P^e = P + bz(e)$ is a parallelotope. Then one can choose a length of the vector e such that $e \in \mathcal{P}_s^*(a)$.*

Proof. Let

$$\mathcal{P}_s^+(a) = \{p \in \mathcal{P}_s(a) : \langle p, e \rangle > 0\}.$$

We show that there is a number $w > 0$ such that $\langle p, e \rangle = w$ for all vectors $p \in \mathcal{P}_s^+(a)$.

Suppose that there are two vectors $p_1, p_2 \in \mathcal{P}_s^+(a)$ such that $\langle p_1, e \rangle \neq \langle p_2, e \rangle$. Since the graph $G(P)$ is connected, there are $p, p' \in \mathcal{P}_s^+(a)$ such that the facets $F(p)$ and $F(p')$ belong to the same 6-belt B , but $\langle p, e \rangle \neq \langle p', e \rangle$. Since $P = P(a)$ is a Voronoi parallelotope, the vector $p - p'$ is a normal vector of a facet of the 6-belt B . Hence the 6-belt B consists of facets $F(\pm p), F(\pm p')$ and $F(\pm(p - p'))$. Since the sum $P + bz(e)$ is a parallelotope, one pair of opposite facets of the belt B belongs to the shadow boundary $\mathcal{F}_e(P)$. Normal vectors of these facets are orthogonal to the vector e . Since $\langle p, e \rangle \neq 0$ and $\langle p', e \rangle \neq 0$, we have $\langle p - p', e \rangle = 0$. This contradicts to the above assertion that $\langle p, e \rangle \neq \langle p', e \rangle$. Therefore, $\langle p, e \rangle = w > 0$ for all $p \in \mathcal{P}_s^+(a)$. Hence, one can choose a length of e such that $\langle p, e \rangle \in \{0, \pm 1\}$ for all $p \in \mathcal{P}_s(a)$. \square

Now we can prove Theorem 2.

Theorem 2. *Let $P(a)$ be an irreducible Voronoi parallelotope, defined in (1), where $\mathcal{P} \supseteq \mathcal{P}(a) \supseteq \mathcal{P}_s(a)$. Let $e \in \mathbb{R}^d$ be a vector. Then one can choose a length of the vector e such that the following assertions are equivalent:*

(i) *Minkowski sum $P(a) + bz(e)$ is a Voronoi parallelotope for any $b \geq 0$, and*

$$P(a) + bz(e) = P(a) + P(a_e) = P(a + a_e);$$

(ii) *$e \in \mathcal{P}_s^*(a)$;*

Proof. Lemma 8 and definition (3) of the set $\mathcal{P}_s^*(a)$ imply the implication (i) \Rightarrow (ii).

We prove implication (ii) \Rightarrow (i). Without loss of generality, we can suppose that $e \in \mathcal{P}_s^*(a) \subseteq \mathcal{P}_e$. By Proposition 2, $P(a) + bz(e) = P(a + a_e)$ is a Voronoi parallelotope. \square

Theorem 2 is a generalization of results for Voronoi polytopes of root lattices D_n , E_6 and E_7 obtained in papers [Gr06a], [DGM14] and [Gr11], respectively.

Note that if the Voronoi parallelotope is reducible, then one can apply Theorem 2 to each component separately.

Theorem 2 has the following important Corollary.

Corollary 1. *If $\mathcal{P}_s^*(a) = \emptyset$, then $P(a) + bz(e)$ is not a parallelotope for any vector e .*

Examples of $P(a)$ with $\mathcal{P}_s^*(a) = \emptyset$ are Voronoi parallelotopes of dual root lattices E_6^* and E_7^* (see [DGM14] and [Gr11]).

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